PROOF OF THE GENESIS IDENTITY:

A NEWLY DISCOVERED MATHEMATICAL THEOREM ENABLING ULTRA-HIGH-SPEED COMPUTATION OF NONLINEAR FIELDS AROUND COMPLEX SHAPES.

by

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The GENESIS Identity is a previously unknown mathematical identity discovered by the author. It is applicable to many areas of nonlinear computational physics (defined below), and offers major advantages in computational time and resources, ease of access by non-specialized users, and the degree of insight it provides into complex physical processes. It has been successfully applied by the author to many different types of fluid-flow and aeroacoustics problems, including both steady-state and time-dependent flows in the subsonic, transonic, supersonic and hypersonic regimes. Unlike the numerous *differential* methods (finite-difference, finite-element and finite-volume techniques) which have been developed in recent years for the solution of nonlinear problems, GENESIS is an *integral*-based formulation which applies the *same* unified process for all types of field, whether they be of the so-called elliptic, hyperbolic or parabolic type. No account needs to be taken of so-called "zones of dependence" which considerably complicate other classes of method for steady-state applications. For time-dependent problems, the same algorithm *automatically* deals with the phenomenon of *retarded time*; in contrast, the existing generation of computational methods needs to make explicit allowance for the finite speed at which disturbances propagate through a medium.

The first generation of integral methods was restricted to the solution of *linear* problems (for example, incompressible, irrotational flow), and involved the numerical evaluation of *surface* integrals representing the influence of surface "source" and/or "dipole" distributions (or, more recently, surface "vorticity" distributions), located on the material surface of the body under investigation. The numerical problem consists of determining these distributions, discretized into "panels," such that the boundary conditions defined by the physical problem were satisfied. Subsequent generations of integral method developed for *nonlinear* problems needed to also evaluate *field* integrals representing the influence of corresponding distributions in the medium external to the body; such field integrations inevitably lead to large computational times, simply because of the high operation-count involved. The GENESIS Identity allows these nonlinear field integrals to be collapsed to *surface* integrals (exactly as for *linear* problems), augmented by a local additive term at each point of the external field. These additional surface terms are then simply interpreted as a modification to the boundary conditions of an "equivalent" *linear* problem. Because the physical problems addressed are truly nonlinear, the process is necessarily *iterative* in nature, but experience shows that the process converges extremely quickly, typically in a few tens of cycles at most, even for highly nonlinear cases involving shock waves or other types of discontinuity in the field. The same underlying linear problem, with a change only in its boundary conditions, is solved at each cycle. The surface integrals themselves thus need to be evaluated only once (again, just as for a linear problem). These combined properties lead to a reduction in computation time, relative to all existing differential methods, of *several orders of magnitude*. The accuracy attainable with the GENESIS approach is, however, at least as good as that of any of these other methods.

The mathematics underlying the GENESIS methodology is complicated, and is not clearly understood even by many of the specialists in integral computational methods. *Any* degree of understanding (or even of belief) by specialists in *differential* computational methods, has been achieved by only a very few individuals who have been sufficiently impressed by the results demonstrated to take the time to study its foundations.

This paper represents an attempt by the author to present, methodically and comprehensively, a definitive proof of the GENESIS Identity and its implications. It is assumed that the reader is familiar with the fundamental elements, theorems and properties of vector fields and standard calculus; any non-standard features are derived here from first principles. The paper is intended for specialists from all areas of computational physics, and minimal reference is made to the writer's own field of fluid mechanics.

GENERAL PROOF OF THE GENESIS IDENTITY

1. STATEMENT OF THE GENESIS IDENTITY

The GENESIS Identity is written in the following general form for an arbitrary, three-dimensional, time-varying or steady-state vector field $\vec{F}(x,y,z)$ defined at a time t in an arbitrary domain Ω with boundary S in an arbitrarily accelerating or stationary frame of reference:

(1) $$T_P\vec{F}_P = \int_\Omega (\nabla_Q \cdot \vec{F}_Q)\vec{K}\, d\Omega_Q + \int_\Omega (\nabla_Q \times \vec{F}_Q) \times \vec{K}\, d\Omega_Q + \int_S (\hat{n}_Q \cdot \vec{F}_Q)\vec{K}\, dS_Q + \int_S (\hat{n}_Q \times \vec{F}_Q) \times \vec{K}\, dS_Q\,.$$

The symbol $\vec{F}_Q \equiv \vec{F}(x_Q, y_Q, z_Q)$ represents the *instantaneous local value* at time t of the spatially varying vector field $\vec{F}$ evaluated at a *running point* Q located at x_Q, y_Q, z_Q in the domain Ω.

The symbol $\vec{F}_P \equiv \vec{F}(x_P, y_P, z_P)$ represents the *instantaneous local value* at time t of that vector field $\vec{F}$ evaluated at a *fixed point* P located at x_P, y_P, z_P which may lie *inside* the domain Ω or *external* to it.

If P lies *inside* domain Ω, the multiplier T_P is equal to unity. If P lies *outside* domain Ω, T_P is zero. These statements include the case where P lies *on* respectively the inner or outer face of the boundary S.

The *vector kernel function* $\vec{K}$ relates to two particular points P and Q and is interpreted as a function defined at running point Q; it depends both upon the coordinates of Q and those of the fixed point P. $\vec{K}$ is defined by the inverse-square vector function of the line vector $\vec{r}_{QP}$ drawn *from* running point Q *to* the fixed point P:

(2) $$\vec{K} \equiv \frac{\hat{r}_{QP}}{4\pi\, r_{QP}^2} \equiv \frac{\vec{r}_{QP}}{4\pi\, r_{QP}^3}\,.$$

Here $\hat{r}_{QP}$ represents the unit vector defined by $\hat{r}_{QP} \equiv \vec{r}_{QP} / r_{QP}$ with r_{QP} the (positive) length of the vector:

$$r_{QP} \equiv \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2}\,.$$

The unit vectors of the cartesian system (x, y, z) employed here are respectively $\hat{i}, \hat{j}, \hat{k}$.

2. PROOF OF THE GENESIS IDENTITY WHEN P LIES OUTSIDE Ω

2.1 Field Differential Operators w.r.t. Q.

We define the *gradient$_Q$ operator* $\nabla_Q(\theta) \equiv \hat{i}\dfrac{\partial\theta}{\partial x_Q} + \hat{j}\dfrac{\partial\theta}{\partial y_Q} + \hat{k}\dfrac{\partial\theta}{\partial z_Q}$ acting on an *arbitrary scalar field* $\theta(x,y,z)$, this gradient being evaluated at the “running” point Q in Ω, w.r.t. changes in the coordinates of Q.

We define the *divergence$_Q$ operator* $\nabla_Q \cdot \vec{H} \equiv \hat{i}\dfrac{\partial H_x}{\partial x_Q} + \hat{j}\dfrac{\partial H_y}{\partial y_Q} + \hat{k}\dfrac{\partial H_z}{\partial z_Q}$ acting on an *arbitrary vector field* $\vec{H}(x,y,z) \equiv \hat{i}H_x(x,y,z) + \hat{j}H_y(x,y,z) + \hat{k}H_z(x,y,z)$, this “divergence” being evaluated at the “running” point Q in Ω, with respect to changes in the coordinates of Q.

We define the *curl$_Q$ operator* $\nabla_Q \times \vec{H} \equiv \hat{i}\left(\dfrac{\partial H_z}{\partial y_Q} - \dfrac{\partial H_y}{\partial z_Q}\right) + \hat{j}\left(\dfrac{\partial H_x}{\partial z_Q} - \dfrac{\partial H_z}{\partial x_Q}\right) + \hat{k}\left(\dfrac{\partial H_y}{\partial x_Q} - \dfrac{\partial H_x}{\partial y_Q}\right)$ acting on an *arbitrary vector field* $\vec{H}(x,y,z) \equiv \hat{i}H_x(x,y,z) + \hat{j}H_y(x,y,z) + \hat{k}H_z(x,y,z)$, this “curl” being evaluated at the “running” point Q in Ω, with respect to changes in the coordinates of Q.

Note: In all these definitions the phrase “Q in Ω” includes the case where Q lies *on* the side of S facing Ω.

2.2 Application of Field Operators to Kernel Function $\vec{K}$

We now consider some differential properties of function $\vec{K}$ evaluated at the running point Q, first noting that:

(3) $$\vec{K} \equiv K_x \hat{i} + K_y \hat{j} + K_z \hat{k} \equiv \frac{x_P - x_Q}{4\pi\ r_{QP}^3} \hat{i} + \frac{y_P - y_Q}{4\pi\ r_{QP}^3} \hat{j} + \frac{z_P - z_Q}{4\pi\ r_{QP}^3} \hat{k}.$$

We define a scalar function G evaluated at Q, this function also being dependent upon the position of point P:

(4) $$G \equiv \frac{1}{4\pi r_{QP}} \equiv \frac{1}{4\pi}\left((x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2\right)^{-1/2}.$$

It is easy to verify that, if $(x_P - x_Q) > 0$ (with y_Q and z_Q held constant), an *increase* in x_Q will *increase* the value of G. In contrast, if $(x_P - x_Q) < 0$, an *increase* in x_Q will *decrease* the value of G. Some simple algebra shows that $\frac{\partial}{\partial x_Q}\left(\frac{1}{4\pi\ r_{QP}}\right) \equiv +\frac{x_P - x_Q}{4\pi r_{QP}^3}$. Extending this result to changes in y_Q and z_Q , we have:

(5) $$\vec{K} \equiv \nabla_Q G \equiv \nabla_Q\left(\frac{1}{4\pi\, r_{QP}}\right)$$

provided the two points P and Q do not coincide, i.e. provided $r_{QP} \neq 0$. This condition will always be satisfied when the fixed point P lies *exterior* to the domain Ω.

Since $\vec{K}$ is the vector gradient of a scalar field, the *curl* of $\vec{K}$ must be zero: $\nabla_Q \times \vec{K} \equiv \nabla_Q \times (\nabla_Q G) \equiv 0$.

In evaluating the *divergence* of $\vec{K}$ at point Q, we have: $4\pi\frac{\partial K_x}{\partial x_Q} = \frac{\partial}{\partial x_Q}\frac{x_P - x_Q}{r_{QP}^3} = \frac{3(x_P - x_Q)^2}{r_{QP}^5} - \frac{1}{r_{QP}^3}$ and similarly $4\pi\frac{\partial K_y}{\partial y_Q} = \frac{3(y_P - y_Q)^2}{r_{QP}^5} - \frac{1}{r_{QP}^3}$ and $4\pi\frac{\partial K_z}{\partial z_Q} = \frac{3(z_P - z_Q)^2}{r_{QP}^5} - \frac{1}{r_{QP}^3}$. Adding these three terms gives $4\pi\nabla_Q \cdot \vec{K} = \frac{3[(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2]}{r_{QP}^5} - \frac{3}{r_{QP}^3} = \frac{3 r_{QP}^2}{r_{QP}^5} - \frac{3}{r_{QP}^3} = 0$.

Collecting these two results, we have:

(6) $$\nabla_Q \cdot \vec{K} = 0 \qquad \textit{and} \qquad \nabla_Q \times \vec{K} \equiv 0.$$

2.3 Application of Field Operators to Fundamental Field "Influence Functions"

2.3.1 Influence of Rotationality of the Field $\vec{F}$

Consider now the vector function $(\nabla_Q \times \vec{F}_Q) \times \vec{K}$ where $\vec{K}$ is the inverse-square vector function defined earlier for some running point Q in the domain Ω and some fixed point P external to Ω, and $\vec{F}_Q$ is the value at Q of an arbitrary vector field $\vec{F}(x,y,z)$, assumed continuous throughout Ω. In the following we interpret this function $(\nabla_Q \times \vec{F}_Q) \times \vec{K}$ as being a vector influence *induced* at the fixed point P by the *inducer field* $\nabla_Q \times \vec{F}_Q$, interpreted as the curl of $\vec{F}(x,y,z)$ evaluated at the running point Q. Decomposing this influence vector into its components parallel to x, y and z, we have:

(7) $$(\nabla_Q \times \vec{F}_Q) \times \vec{K} \equiv \hat{i}\,[((\nabla_Q \times \vec{F}_Q) \times \vec{K}) \cdot \hat{i}\,] + \hat{j}\,[((\nabla_Q \times \vec{F}_Q) \times \vec{K}) \cdot \hat{j}\,] + \hat{k}\,[((\nabla_Q \times \vec{F}_Q) \times \vec{K}) \cdot \hat{k}\,].$$

Now from the standard definition of the vector product $\vec{A}\times\vec{B}$ of *any* two vectors $\vec{A}$ and $\vec{B}$, we can write the *x*-component of $\vec{A}\times\vec{B}$ in the form: $\hat{i}\cdot(\vec{A}\times\vec{B}) = -B_y A_z + B_z A_y = -B_y(\hat{k}\cdot\vec{A}) + B_z(\hat{j}\cdot\vec{A})$. With $\nabla_Q\times\vec{F}_Q$ as vector $\vec{A}$ and $\vec{K}$ as vector $\vec{B}$, we thus have for the *x*-component of the vector product on the l.h.s. of (7):

$$[(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\cdot\hat{i} \equiv -K_y[\hat{k}\cdot(\nabla_Q\times\vec{F}_Q)] + K_z[\hat{j}\cdot(\nabla_Q\times\vec{F}_Q)].$$

From the definition of the curl_Q operator in Sec. 2.1 we then have:

$$\begin{aligned}
&[(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\cdot\hat{i} \equiv -K_y[\hat{k}\cdot(\nabla_Q\times\vec{F}_Q)] + K_z[\hat{j}\cdot(\nabla_Q\times\vec{F}_Q)] \\
&\equiv K_x[0] \;-\; K_y\left[\frac{\partial F_y}{\partial x_Q} - \frac{\partial F_x}{\partial y_Q}\right] \;+\; K_z\left[\frac{\partial F_x}{\partial z_Q} - \frac{\partial F_z}{\partial x_Q}\right] \\
&\equiv K_x\left[\frac{\partial F_x}{\partial x_Q} - \frac{\partial F_x}{\partial x_Q}\right] \;-\; K_y\left[\frac{\partial F_y}{\partial x_Q} - \frac{\partial F_x}{\partial y_Q}\right] \;+\; K_z\left[\frac{\partial F_x}{\partial z_Q} - \frac{\partial F_z}{\partial x_Q}\right] \\
&\equiv \;-\left[K_x\frac{\partial F_x}{\partial x_Q} \;+\; K_y\frac{\partial F_y}{\partial x_Q} \;+\; K_z\frac{\partial F_z}{\partial x_Q}\right] \;+\; \left[K_x\frac{\partial F_x}{\partial x_Q} + K_y\frac{\partial F_x}{\partial y_Q} + K_z\frac{\partial F_x}{\partial z_Q}\right] \\
&\equiv \left[-\left(\frac{\partial(F_x K_x)}{\partial x_Q} \;+\; \frac{\partial(F_y K_y)}{\partial x_Q} \;+\; \frac{\partial(F_z K_z)}{\partial x_Q}\right) + \left(F_x\frac{\partial K_x}{\partial x_Q} \;+\; F_y\frac{\partial K_y}{\partial x_Q} \;+\; F_z\frac{\partial K_z}{\partial x_Q}\right)\right] + \vec{K}\cdot\nabla_Q F_x\,.
\end{aligned}$$

Now from the standard definition of the scalar product $\vec{A}\cdot\vec{B}$ of *any* two vectors $\vec{A}$ and $\vec{B}$ we have: $A_x B_x + A_y B_y + A_z B_z = \vec{A}\cdot\vec{B}$. Using this in the final r.h.s. of the previous identity, we thus have:

(8) $$[(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\cdot\hat{i} \;\equiv\; -\frac{\partial(\vec{K}\cdot\vec{F}_Q)}{\partial x_Q} \;+\; \vec{F}_Q\cdot\frac{\partial\vec{K}}{\partial x_Q} \;+\; \vec{K}\cdot\nabla_Q F_x\,.$$

Now $\vec{K}\cdot\vec{F}_Q$ is a scalar field evaluated at the running point *Q*, and for any scalar field $\theta(x,y,z)$ we can write $\partial\theta/\partial x \equiv \nabla\cdot(\theta\hat{i})$. Thus for the first term on the r.h.s. of (8) we have $-\partial(\vec{K}\cdot\vec{F}_Q)/\partial x_Q \equiv -\nabla_Q\cdot[(\vec{K}\cdot\vec{F}_Q)\hat{i}]$. Secondly we note for the final term on the r.h.s. of (8) that

$$\begin{aligned}
&\vec{K}\cdot\nabla_Q F_x \equiv K_x\frac{\partial F_x}{\partial x_Q} + K_y\frac{\partial F_x}{\partial y_Q} + K_z\frac{\partial F_x}{\partial z_Q} \equiv \left(\frac{\partial F_x K_x}{\partial x_Q} + \frac{\partial F_x K_y}{\partial y_Q} + \frac{\partial F_x K_z}{\partial z_Q}\right) - \left(F_x\frac{\partial K_x}{\partial x_Q} + F_x\frac{\partial K_y}{\partial y_Q} + F_x\frac{\partial K_z}{\partial z_Q}\right) \\
&\equiv \nabla_Q\cdot[(\vec{F}_Q\cdot\hat{i})\vec{K}] \;-\; F_x\nabla_Q\cdot\vec{K} \;\equiv\; \nabla_Q\cdot[(\vec{F}_Q\cdot\hat{i})\vec{K}]
\end{aligned}$$

where we have used the result (6) derived earlier stating that $\nabla_Q\cdot\vec{K}\equiv 0$ when P does not coincide with Q. Thus we can finally write for the *x*-component of the vector field $(\nabla_Q\times\vec{F}_Q)\times\vec{K}$ *induced* at the fixed point P by the *inducer field* $\nabla_Q\times\vec{F}_Q$ evaluated at the running point Q:

(9a) $$[(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\cdot\hat{i} \;\equiv\; \nabla_Q\cdot[(\vec{F}_Q\cdot\hat{i})\vec{K} - (\vec{K}\cdot\vec{F}_Q)\hat{i}] \;+\; \vec{F}_Q\cdot(\partial\vec{K}/\partial x_Q)$$

and similarly for the y- and z-components:

(9b) $$[(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\cdot\hat{j} \;\equiv\; \nabla_Q\cdot[(\vec{F}_Q\cdot\hat{j})\vec{K} - (\vec{K}\cdot\vec{F}_Q)\hat{j}] \;+\; \vec{F}_Q\cdot(\partial\vec{K}/\partial y_Q)$$

(9c) $$[(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\cdot\hat{k} \;\equiv\; \nabla_Q\cdot[(\vec{F}_Q\cdot\hat{k})\vec{K} - (\vec{K}\cdot\vec{F}_Q)\hat{k}] \;+\; \vec{F}_Q\cdot(\partial\vec{K}/\partial z_Q)\,.$$

2.3.2 *Influence of Divergence of the Field* $\vec{F}$

Consider now the vector function $(\nabla_Q \cdot \vec{F}_Q)\vec{K}$ where $\vec{K}$ and $\vec{F}_Q$ are as defined in Sec. 2.3.1. In the following we interpret this function $(\nabla_Q \cdot \vec{F}_Q)\vec{K}$ as being a vector influence *induced* <u>at the fixed point</u> P by the *inducer field* $\nabla_Q \cdot \vec{F}_Q$, interpreted as the divergence of $\vec{F}(x,y,z)$ <u>evaluated at the running point</u> Q. Decomposing this influence vector into its components parallel to *x*, *y* and *z*, we have:

$$(\nabla_Q \cdot \vec{F}_Q)\vec{K} \equiv (\nabla_Q \cdot \vec{F}_Q)K_x\,\hat{i} + (\nabla_Q \cdot \vec{F}_Q)K_y\,\hat{j} + (\nabla_Q \cdot \vec{F}_Q)K_z\,\hat{k}\,.$$

Considering only the *x*-component of this field induced at P, and noting that for the combination of any scalar field θ and any vector field $\vec{H}$ we have the vector identity $(\nabla \cdot \vec{H})\theta \equiv \nabla_Q \cdot (\theta\vec{H}) - \vec{H}\cdot(\nabla\theta)$, we obtain:

(10) $$(\nabla_Q \cdot \vec{F}_Q)\vec{K}\cdot\hat{i} \equiv (\nabla_Q \cdot \vec{F}_Q)K_x \equiv \nabla_Q \cdot (K_x\vec{F}_Q) - \vec{F}_Q \cdot \nabla_Q K_x \quad \text{where} \quad \nabla_Q K_x \equiv \frac{\partial K_x}{\partial x_Q}\hat{i} + \frac{\partial K_x}{\partial y_Q}\hat{j} + \frac{\partial K_x}{\partial z_Q}\hat{k}\,.$$

But since we know from (6) that $\nabla_Q \times \vec{K} \equiv 0$ we have $\dfrac{\partial K_x}{\partial y_Q} = \dfrac{\partial K_y}{\partial x_Q}$ and $\dfrac{\partial K_x}{\partial z_Q} = \dfrac{\partial K_z}{\partial x_Q}$ so we can write:

$$\nabla_Q K_x \equiv \frac{\partial K_x}{\partial x_Q}\hat{i} + \frac{\partial K_y}{\partial x_Q}\hat{j} + \frac{\partial K_z}{\partial x_Q}\hat{k} \equiv \frac{\partial \vec{K}}{\partial x_Q}\,.$$

Inserting these results in (10) then gives for the *x*-component of $(\nabla_Q \cdot \vec{F}_Q)\vec{K}$:

(11a) $$(\nabla_Q \cdot \vec{F}_Q)\vec{K}\cdot\hat{i} \equiv \nabla_Q \cdot (K_x\vec{F}_Q) - \vec{F}_Q \cdot (\partial\vec{K}/\partial x_Q)$$

and similarly for the y- and z-components:

(11b) $$(\nabla_Q \cdot \vec{F}_Q)\vec{K}\cdot\hat{j} \equiv \nabla_Q \cdot (K_y\vec{F}_Q) - \vec{F}_Q \cdot (\partial\vec{K}/\partial y_Q)$$

(11c) $$(\nabla_Q \cdot \vec{F}_Q)\vec{K}\cdot\hat{k} \equiv \nabla_Q \cdot (K_z\vec{F}_Q) - \vec{F}_Q \cdot (\partial\vec{K}/\partial z_Q)\,.$$

2.3.3 *Influence of Combined Rotationality and Divergence of the Field* $\vec{F}$

Adding the identities (9a) and (11a) for the *x*-components of the rotationality-related and divergence-related fields induced at the fixed point P external to Ω by the inducers located at running point Q in Ω, noting that upon addition the term $\vec{F}_Q \cdot \dfrac{\partial\vec{K}}{\partial x_Q}$ then vanishes, gives:

(12) $$[(\nabla_Q \cdot \vec{F}_Q)\vec{K} + (\nabla_Q \times \vec{F}_Q)\times\vec{K}]\cdot\hat{i} \equiv \nabla_Q \cdot [(K_x\vec{F}_Q) + (\vec{F}_Q\cdot\hat{i})\vec{K} - (\vec{K}\cdot\vec{F}_Q)\hat{i}\,]\,.$$

Now the r.h.s. of this (scalar) identity is purely in divergence form, so we can apply <u>Gauss' divergence theorem</u>; for a vector field $\vec{H}(x,y,z)$ in a domain Ω with boundary S this is generally expressed in the form:

(13) $$\iiint_\Omega (\nabla\cdot\vec{H})\,d\Omega \equiv \iint_S (\hat{n}'\cdot\vec{H})\,dS\,.$$

Here $\hat{n}'$ is the unit normal at the surface S, pointing *outwards* from Ω. In practical problems of integro-differential calculus, the domain Ω is usually the region of space lying *external* to some body at which boundary conditions are to be applied. In such cases it is usual to define the unit surface normal $\hat{n}$ pointing *outwards* from the body, and therefore pointing *into* Ω. Thus in Gauss' divergence theorem we here take the unit normal $\hat{n}_Q = -\hat{n}'$, defined at the running point Q on the boundary S, to point *into* the domain Ω.

We thus have for an arbitrary vector field $\vec{H}(x,y,z)$:

(14) $$\iiint_{\Omega}(\nabla\cdot\vec{H})\,d\Omega \equiv -\iint_{S}(\hat{n}_Q\cdot\vec{H})\,dS\,.$$

Integrating (12) over all points Q in Ω, and applying (14) to the r.h.s., we have at the *external* fixed point P:

(15) $$\begin{aligned}&\iiint_{\Omega}[(\nabla_Q\cdot\vec{F}_Q)\vec{K}+(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\cdot\hat{i}\,d\Omega \equiv \iiint_{\Omega}\nabla_Q\cdot[(K_x\vec{F})+(\vec{F}_Q\cdot\hat{i})\vec{K}-(\vec{K}\cdot\vec{F}_Q)\hat{i}\,]\,d\Omega\\ &\equiv -\iint_{S}\hat{n}_Q\cdot[(K_x\vec{F})+(\vec{F}_Q\cdot\hat{i})\vec{K}-(\vec{K}\cdot\vec{F}_Q)\hat{i}\,]dS \equiv -\iint_{S}[(\hat{n}_Q\cdot\vec{F})K_x+(\vec{F}_Q\cdot\hat{i})\vec{K}\cdot\hat{n}_Q-(\vec{K}\cdot\vec{F}_Q)\hat{i}\cdot\hat{n}_Q\,]dS\,.\end{aligned}$$

Now a standard vector identity for a triple vector product of three arbitrary vectors $\vec{H},\vec{L},\vec{M}$ states that: $(\vec{H}\times\vec{L})\times\vec{M}\equiv(\vec{H}\cdot\vec{M})\vec{L}-(\vec{M}\cdot\vec{L})\vec{H}$. Applying this result to the combination of the last two terms in the surface integrand in (15) thus gives:

(16) $$(\vec{F}_Q\cdot\hat{i})\vec{K}\cdot\hat{n}_Q-(\vec{K}\cdot\vec{F}_Q)\hat{i}\cdot\hat{n}_Q \equiv \hat{i}\cdot[(\vec{K}\cdot\hat{n}_Q)\vec{F}_Q-(\vec{K}\cdot\vec{F}_Q)\,\hat{n}_Q] \equiv \hat{i}\cdot[(\hat{n}_Q\times\vec{F}_Q)\times\vec{K}]\,.$$

Inserting this result into the identity (15) for the *x*-component of the overall induced vector field, and combining with the corresponding results for the *y*- and *z*-components, we finally obtain at the fixed point P external to Ω the identity:

(17) $$\iiint_{\Omega}[(\nabla_Q\cdot\vec{F}_Q)\vec{K}+(\nabla_Q\times\vec{F}_Q)\times\vec{K}]\,d\Omega \equiv -\iint_{S}[(\hat{n}_Q\cdot\vec{F})\,\vec{K}+(\hat{n}_Q\times\vec{F}_Q)\times\vec{K}]\,dS\,.$$

This completes the proof of the GENESIS Identity for the case where the fixed point P appearing in the definition of the vector kernel function $\vec{K}$ lies *external* to the domain Ω (and thus never coincides with the running point Q). This result <u>does include</u> the case where the point P (external to Ω) approaches arbitrarily closely to the boundary S and in the limit is located on the outer face of that boundary (i.e. the face pointing *outwards* from Ω).

3. PROOF OF THE GENESIS IDENTITY WHEN P LIES INSIDE Ω

For the case where the fixed point P lies in the *interior* of the domain Ω with boundary S, we employ a proof which is partly analytical and partly intuitive. First we construct a small sphere of radius b centered on the point P. We designate the interior of this sphere as the small, spherical domain ε with spherical surface χ. The point P now lies in the *interior* of ε but *exterior* to the remaining volume $\Omega - \varepsilon$ whose boundary consists of the original boundary S plus this new boundary χ. The vector field $\vec{F}(x,y,z)$ is now defined as a continuous vector function both in the domain $\Omega - \varepsilon$ and in the domain ε, and is continuous across the interface boundary χ. As the radius b becomes vanishingly small, the field $\vec{F}(x,y,z)$ becomes progressively *closer* throughout the interior of the infinitesimal domain ε to the value $\vec{F}_P$ of the field $\vec{F}(x,y,z)$ at the midpoint P of the sphere. Similarly, the functions $\nabla_Q \cdot \vec{F}_Q$ and $\nabla_Q \times \vec{F}_Q$ become progressively closer, throughout the interior of ε, to their values at P.

We can now consider the overall "influence" at the point P as the sum of two distinct contributions: the first from the volume $\Omega - \varepsilon$ (noting that P lies *exterior* to this volume so that the "exterior" proof given in Section 2 above applies here); and the second from the small sphere ε surrounding P. Considering running points Q ranging over the entire domain Ω (including the small sphere ε) we can write at the point P:

(18)
$$\int_\Omega (\nabla_Q \cdot \vec{F}_Q)\vec{K}\, d\Omega + \int_\Omega (\nabla_Q \times \vec{F}_Q) \times \vec{K}\, d\Omega \equiv \int_{\Omega-\varepsilon} (\nabla_Q \cdot \vec{F}_Q)\vec{K}\, d\Omega + \int_{\Omega-\varepsilon} (\nabla_Q \times \vec{F}_Q) \times \vec{K}\, d\Omega$$
$$+ \int_\varepsilon (\nabla_Q \cdot \vec{F}_Q)\vec{K}\, d\varepsilon + \int_\varepsilon (\nabla_Q \times \vec{F}_Q) \times \vec{K}\, d\varepsilon\,.$$

Although the kernel function $\vec{K}$ as defined by (2) becomes arbitrarily large within the small sphere ε (i.e. as the line vector r_{QP} becomes arbitrarily short as the point P approaches the point Q), we observe that this function is *antisymmetric* about the point P: Within this sphere ε, for every point Q with a line vector $\vec{r}_{QP}$ relative to P, there is a separate point Q' with a line vector $-\vec{r}_{QP}$, this second point being located on the opposite side of P. Since in the limit of small radius b the functions $\nabla_Q \cdot \vec{F}_Q$ and $\nabla_Q \times \vec{F}_Q$ approach the same values at Q and Q', their combined "influence" must vanish at the point P. This intuitive, general argument corresponds directly, for example, to the familiar, specific result that the gravitational field induced at the center of a spherical mass by its (spherically symmetric) mass distribution, is zero.

On the basis of this argument, we can therefore ignore the last two integrals written in the above identity (18), i.e. those evaluated over the sphere ε in the limit of a vanishingly small radius b. We then have:

(19)
$$\int_\Omega (\nabla_Q \cdot \vec{F}_Q)\vec{K}\, d\Omega + \int_\Omega (\nabla_Q \times \vec{F}_Q) \times \vec{K}\, d\Omega \equiv \int_{\Omega-\varepsilon} (\nabla_Q \cdot \vec{F}_Q)\vec{K}\, d\Omega + \int_{\Omega-\varepsilon} (\nabla_Q \times \vec{F}_Q) \times \vec{K}\, d\Omega\,.$$

We can now apply the GENESIS Identity (17) derived earlier (for the case where P lies *external* to the "influencing" domain) to the domain $\Omega - \varepsilon$ appearing on the r.h.s. of (19). Noting that the overall boundary of this domain is $S + \chi$ and writing the contributions separately for the two distinct parts of this overall boundary (with $\hat{n}_Q$ on each sub-boundary pointing *into* the reduced domain $\Omega - \varepsilon$), we have:

(20)
$$\iiint_{\Omega-\varepsilon} [(\nabla_Q \cdot \vec{F}_Q)\vec{K} + (\nabla_Q \times \vec{F}_Q) \times \vec{K}]d\Omega \equiv -\iint_S [(\hat{n}_Q \cdot \vec{F}_Q)\vec{K} + (\hat{n}_Q \times \vec{F}_Q) \times \vec{K}]dS$$
$$-\iint_\chi [(\hat{n}_Q \cdot \vec{F}_Q)\vec{K} + (\hat{n}_Q \times \vec{F}_Q) \times \vec{K}]d\chi\,.$$

Now a standard vector identity for any three arbitrary vectors $\vec{H}, \vec{L}, \vec{M}$ states that: $(\vec{H}\cdot\vec{L})\vec{M} + (\vec{H}\times\vec{L})\times\vec{M} \equiv (\vec{H}\cdot\vec{M})\vec{L} + (\vec{H}\times\vec{M})\times\vec{L}$. Applying this result to the final surface integrand in (20) gives: $(\hat{n}_Q\cdot\vec{F}_Q)\vec{K} + (\hat{n}_Q\times\vec{F}_Q)\times\vec{K} \equiv (\hat{n}_Q\cdot\vec{K})\vec{F}_Q + (\hat{n}_Q\times\vec{K})\times\vec{F}_Q$. Now on the boundary χ of the small sphere ε of radius b, the unit normal $\hat{n}_Q$ at any point Q on that boundary is defined to point *into* $\Omega-\varepsilon$ (and therefore *out of* ε). Also, the line vector $\vec{r}_{QP}$ is drawn *from* Q (here on the surface of the sphere) *to* P (here at the center of the sphere). Thus for the unit vectors we have $\hat{n}_Q = -\hat{r}_{QP}$ and using the above statement for the value of the vector $\vec{F}_Q$ on the surface of the small sphere ε we immediately obtain for a very small radius b:

(21) $$(\hat{n}_Q\cdot\vec{K})\vec{F}_Q \equiv -\frac{\vec{F}_P}{4\pi b^2} \quad \text{and} \quad (\hat{n}_Q\times\vec{K})\times\vec{F}_Q \equiv 0\,.$$

Thus the integral over the infinitesimal surface χ in the above identity (20) becomes:

$$-\iint_\chi [(\hat{n}_Q\cdot\vec{F}_Q)\vec{K} + (\hat{n}_Q\times\vec{F}_Q)\times\vec{K}]d\chi \equiv \int_\chi \frac{\vec{F}_P}{4\pi b^2}d\chi \equiv \frac{\vec{F}_P}{4\pi b^2}\int_\chi d\chi \equiv \vec{F}_P\,.$$

Collecting these results, inserting in the above identity (20) and rearranging, we finally obtain:

(22) $$\vec{F}_P \equiv \iiint_\Omega [(\nabla_Q\cdot\vec{F}_Q)\vec{K} + (\nabla_Q\times\vec{F}_Q)\times\vec{K}]d\Omega + \iint_S [(\hat{n}_Q\cdot\vec{F}_Q)\vec{K} + (\hat{n}_Q\times\vec{F}_Q)\times\vec{K}]dS\,.$$

This completes the proof of the GENESIS Identity for the case where the point P lies in the *interior* of the domain Ω in which the fields $\nabla_Q\cdot\vec{F}_Q = 0$ and $\nabla_Q\times\vec{F}_Q = 0$ are defined.

A trivial extension of this approach, not discussed here, gives the modified result for the case where the point P lies at a point *on* the boundary S where the boundary is either smooth or contains an angular discontinuity. However, it may be noted that in practical numerical schemes ("panel methods") based either on the above generalized nonlinear identity or on its linear (Laplace) subset in which the field integrals vanish, this special treatment for "boundary points" may be avoided. This is achieved by simply computing *analytically*, for a finite part of the surface S (a "panel"), the analytical *limit* of the general "field" expression (i.e. for P in the interior of Ω) as the point P approaches that region of the surface (i.e. *approaches* the midpoint of a surface panel). This type of reasoning completely removes the need to consider the (Cauchy) Principal Value of the integrals involved.

4. INTERPRETATION OF INDUCER FIELDS

4.1 Gauss' Divergence Theorem Revisited

In the derivation of the GENESIS Identity we quoted *Gauss' Divergence Theorem*, one of the most frequently applied and well-established theorems in mathematical physics. The proof of this theorem is somewhat intuitive and it is often considered almost as an *empirical* statement relating the *total flux* of some vector field through a closed surface to the total amount of "stuff" contained in the volume bounded by that surface. The "stuff" in question is usually considered as a *physical entity* such as electric charge or physical mass. We here consider Gauss' Divergence Theorem to be a *mathematical identity*. Applying this identity [eqn. (14)] to the arbitrary vector field $\vec{F}(x,y,z)$ defined in an arbitrary domain Ω with closed boundary S we have:

(23)
$$\iiint_\Omega (\nabla_Q \cdot \vec{F}_Q)\, d\Omega \equiv -\iint_S (\hat{n}_Q \cdot \vec{F}_Q)\, dS$$

in which for consistency we have again chosen the unit normal $\hat{n}_Q$ at the point Q on the boundary S to point *into* Ω. We may introduce the symbol $\Sigma_Q \equiv \nabla_Q \cdot \vec{F}_Q$ to represent the field intensity (quantity per unit *volume*) of the "stuff" appearing in the integrand on the l.h.s. of (23), and the symbol $\sigma_Q = -\hat{n}_Q \cdot \vec{F}_Q$ to represent the surface intensity (quantity per unit *area*) of the "stuff" appearing in the integrand on the r.h.s. This is an alternative to interpreting $\hat{n}_Q \cdot \vec{F}_Q$ as the flux density of the vector field through the surface S. With this interpretation we immediately observe that the *total amount* of field "source stuff" $\iiint_\Omega \Sigma_Q\, d\Omega$ contained in the domain Ω is *exactly equal* to the *total amount* of surface "source stuff" $\iint_S \sigma_Q\, dS$ contained on its boundary S.

A "rotational" counterpart of Gauss' theorem may be derived from (23), though this is not so well known as the above "divergence" form. For this we consider the total amount of "rotationality" of the vector $\vec{F}(x,y,z)$ contained in the volume Ω, namely: $\iiint_\Omega (\nabla_Q \times \vec{F}_Q)\, d\Omega$. We observe that:

$$\begin{aligned}\nabla_Q \times \vec{F}_Q &\equiv \hat{i}\left(\frac{\partial F_z}{\partial y_Q} - \frac{\partial F_y}{\partial z_Q}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z_Q} - \frac{\partial F_z}{\partial x_Q}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x_Q} - \frac{\partial F_x}{\partial y_Q}\right)\\ &\equiv \hat{i}\,\nabla_Q \cdot (0\hat{i} + F_z\hat{j} - F_y\hat{k}) + \hat{j}\,\nabla_Q \cdot (-F_z\hat{i} + 0\hat{j} + F_x\hat{k}) + \hat{k}\,\nabla_Q \cdot (F_y\hat{i} - F_x\hat{j} + 0\hat{k})\end{aligned}$$

so that applying (23) to each of the three "divergence" forms in the final member of this identity, we have:

$$\begin{aligned}\iiint_\Omega (\nabla_Q \times \vec{F})\, d\Omega &\equiv \hat{i}\iiint_\Omega [\nabla_Q \cdot (0\hat{i} + F_z\hat{j} - F_y\hat{k})]\, d\Omega + \hat{j}\iiint_\Omega \nabla_Q \cdot [(-F_z\hat{i} + 0\hat{j} + F_x\hat{k})]\, d\Omega + \hat{k}\iiint_\Omega \nabla_Q \cdot [(F_y\hat{i} - F_x\hat{j} + 0\hat{k})]\, d\Omega\\ &\equiv -\hat{i}\iint_S \hat{n}_Q \cdot (0\hat{i} + F_z\hat{j} - F_y\hat{k})\, dS - \hat{j}\iint_S \hat{n}_Q \cdot (-F_z\hat{i} + 0\hat{j} + F_x\hat{k})\, dS - \hat{k}\iint_\Omega \hat{n}_Q \cdot (F_y\hat{i} - F_x\hat{j} + 0\hat{k})\, dS\\ &\equiv -\hat{i}\iint_S (n_y F_z - n_z F_y)\, dS - \hat{j}\iint_S (-n_x F_z\hat{i} + n_z F_x)\, dS - \hat{k}\iint_\Omega (n_x F_y - n_y F_x)\, dS\\ &\equiv -\iint_S [(n_y F_z - n_z F_y)\hat{i} + (-n_x F_z\hat{i} + n_z F_x)\hat{j} + (n_x F_y - n_y F_x)\hat{k}]\, dS\end{aligned}$$

or finally:

(24)
$$\iiint_\Omega (\nabla_Q \times \vec{F}_Q)\, d\Omega \equiv -\iint_S (\hat{n}_Q \times \vec{F}_Q)\ dS\ .$$

We can introduce the symbol $\vec{\Gamma}_Q \equiv \nabla_Q \times \vec{F}_Q$ to represent the field (vector) intensity (quantity per unit *volume*) of the "rotational stuff" appearing in the integrand on the l.h.s. of (24), and the symbol $\vec{\omega}_Q = -\hat{n}_Q \times \vec{F}_Q$ to represent the surface (vector) intensity (quantity per unit *area*) of the "rotational stuff" appearing in the integrand on the r.h.s. With this interpretation we immediately observe that the *total amount* of field "rotational stuff" $\iiint_\Omega \vec{\Gamma}_Q \, d\Omega$ contained in the domain Ω is *exactly equal* to the *total amount* of surface "rotational stuff" $\iint_S \vec{\omega}_Q \, dS$ contained on its boundary S.

The two theorems in (23) and (24) provide no information on the field *induced* by the "stuff" in the domain Ω, except in some cases which are extremely simple geometrically. Consider for example the case of a spherical domain Ω with constant radius R. Consider a distribution of "source stuff" Σ which is spherically symmetric within Ω, i.e. $\Sigma = \nabla \cdot \vec{F}$ is a function of only the radial distance r from the center of the sphere. Obviously the corresponding vector $\vec{F}$ must be a purely radial vector: $\vec{F} = F(r)\,\hat{r}$. Now in spherical coordinates the divergence of a radial vector is defined by the identity: $\nabla \cdot \vec{F} = \dfrac{1}{r^2}\dfrac{d}{dr}(r^2 F)$ so we have at a radius r: $F(r) = \dfrac{1}{r^2}\int_0^r r^2 \Sigma \, dr$. Suppose for example that the spherically symmetric function $\Sigma(r)$ takes the form: $\Sigma = \Sigma_p r^p$, where the constant p can take any fractional or integer value. Any desired form for $\Sigma(r)$ can be constructed as the linear sum of a number of such functions with different values of the index p (for example, a *uniform* distribution is formed by a single function with the index $p = 0$). For a single function with index p we have at the radius r: $F_p(r) = \dfrac{\Sigma_p}{r^2}\int_0^r r^{p+2}\, dr = \dfrac{\Sigma_p}{p+3}\, r^{p+1}$. Now the total amount of "stuff" $\Sigma_{tot\,r}$ contained in the region of the sphere between $r = 0$ and radius r is given by: $\Sigma_{tot\,r} = \int_0^r (\Sigma_p r^p) 4\pi r^2 \, dr =$ $4\pi\, \Sigma_p \int_0^r r^{p+2}\, dr = \dfrac{4\pi \Sigma_p}{p+3}\, r^{p+3}$ so that combining these results and using the symbol $\vec{K}$ introduced in (2), we can write at radius r: $\vec{F}_p = F_p(r)\,\hat{r} = \dfrac{\Sigma_{tot\,r}}{4\pi\, r^2}\,\hat{r} = \Sigma_{tot\,r}\,\vec{K}$. Thus at a radius $r < R$ the field $F_p(r)$ is identical to that induced by a "*point* inducer" whose strength is equal to the total amount of *field* inducer contained inside the sphere of radius r. The contribution at the radius r from the remaining field source located at a greater radius (between r and R) is zero (a result familiar from gravitational theory). At any radius $r > R$ the field $F_p(r)$ is the same as that induced by "surface stuff" of surface density $\sigma = -\hat{n} \cdot \vec{F}_p = \Sigma_{tot\,R} / (4\pi\, R^2)$ on the surface of the sphere of radius R. The total amount of this "surface stuff" is $\sigma_{tot\,R} = 4\pi R^2 \sigma = \Sigma_{tot\,R}$ which is also equal to the strength of the equivalent point inducer located at the center of the sphere. An analogous result, not developed here, can be obtained in similar fashion for the "rotational" counterpart given by (24). Thus for this special, spherically symmetric case, Gauss' theorem (23, 24) replicates the result of the GENESIS Identity (22). For more general geometries, no such result can be obtained purely from Gauss' theorem, and the GENESIS Identity (22) must instead be used. It is worthy of note that for Gauss' Theorems the *nature* of the field stuff and surface stuff on respectively the l.h.s. and r.h.s. of (23) is the same (i.e. "source stuff"), and that on the l.h.s. and r.h.s. of (24) is also the same (i.e. in this case "rotational stuff"). However, in the case of the GENESIS Identity (23), *both* types of stuff ("source" *and* "rotationality") in general appear in the surface integrals, even when the field integrals contain *only one type* of "stuff" [for example, only "source stuff" if the vector $\vec{F}(x,y,z,)$ is irrotational so that $\nabla_Q \times \vec{F} \equiv 0$)]. This observation will be pursued further below.

4.2 Field Inducers and Equivalent Surface Inducers in the GENESIS Identity

We have seen that for *any* vector field $\vec{F}(x,y,z)$ in *any* domain Ω whose boundary is the closed surface S with the local unit normal $\hat{n}_Q$ at any point Q on S defined to point *into* Ω, we can write the following relationship ("GENESIS Identity") which is valid at *any* point P:

(25) $$T_P\,\vec{F}_P \;\equiv\; \iiint_\Omega [(\nabla_Q\cdot\vec{F}_Q)\vec{K} + (\nabla_Q\times\vec{F}_Q)\times\vec{K}]d\Omega \;+\; \iint_S [(\hat{n}_Q\cdot\vec{F}_Q)\,\vec{K} + (\hat{n}_Q\times\vec{F}_Q)\times\vec{K}]dS\,.$$

We saw that the multiplier T_P takes the value $T_P = 1$ if the point P lies in the interior of Ω and takes the value $T_P = 0$ when P lies external to Ω. The cases where P lies *on* a smooth or non-smooth part of the boundary S were discussed briefly earlier in Section 3.

The *mathematical identity* (25) applies to *any* vector field, including such familiar examples as the fields associated with *gravity*, *electromagnetism*, *electrodynamics*, *magnetohydrodynamics*, or *fluid flow*. In the interests of generality, the particular *physics* associated with these fields will not be discussed in detail here; the case of fluid flow and the associated thermodynamics have been discussed extensively elsewhere by the author. There is just one essential point to understand here: namely, that the *physics* of a particular situation allows a *local value* to be established for the intensity of the field inducers. For example, for the case of a steady-state flow of an ideal fluid in the subsonic, transonic, supersonic or hypersonic regime, the equations governing *mass continuity*, *momentum* and/or *energy* allow a value for the divergence $\nabla_Q\cdot\vec{F}_Q$ at a point Q to be established purely in terms of the local flowspeed $V = \left|\vec{V}\right|$ and its spatial (vector) derivative $\nabla_Q\vec{V}$. For different physics, for example if a chemical reaction is present, the governing equations are modified in some known way, thus giving a different local *value* for $\nabla_Q\cdot\vec{F}_Q$. In an *integral* formulation for the computation of a compressible fluid flow, the computational variables are the *velocity vector* $\vec{V}$, its *divergence* $\nabla_Q\cdot\vec{V}$ and *curl* $\nabla_Q\times\vec{V}$, the latter two values being established at each point of the field from the governing physics, in terms of the current estimate of the velocity field $\vec{V}$. The usual thermodynamic variables (pressure, density, temperature, entropy, etc.) do not feature during the computation itself; they are used in the preliminary *analysis* (i.e. *before* the computer code is established) and if required their values can be obtained trivially as a by-product of the computation. Similar arguments apply for the values of $\nabla_Q\cdot\vec{F}_Q$ and $\nabla_Q\times\vec{F}_Q$ in other branches of physics.

A *field* inducer of intensity $\nabla_Q\cdot\vec{F}_Q$ in an elemental volume $d\Omega$ located at a point Q (the value $\nabla_Q\cdot\vec{F}_Q$ being obtained from the governing *physics*) may be considered as a *field source* of strength $(\nabla_Q\cdot\vec{F}_Q)d\Omega$ which (considered as an isolated point source) *induces* a <u>radial</u> field. We introduce the symbol Σ to represent the *physically-defined* field source intensity (i.e. we write $\Sigma_Q \equiv \nabla_Q\cdot\vec{F}_Q$). This symbol could for example represent an *electric charge density* in an inhomogeneous electric field, or *mass density* in the case of gravitation, or a *fluid source density* in the case of compressible fluid flow (subsonic or supersonic), etc. The elemental contribution to the overall vector field induced at the point P may now be written as:

$d\vec{F}_P = (\Sigma_Q\,d\Omega)\,\vec{K} \equiv \dfrac{(\Sigma_Q\,d\Omega)\,\hat{r}_{QP}}{4\pi\,r_{QP}^2}$ which may be recognized as the familiar *inverse-square law* of gravity or *Coulomb's law* in electrostatics. Correspondingly we can introduce the symbol $\vec{\Gamma}_Q \equiv \nabla_Q\times\vec{F}_Q$. This symbol could for example represent the *electric current density* vector at a point Q in an electromagnetic field or the *vorticity* in a rotational fluid flow. In this case we have at a point P: $d\vec{F}_P = (\vec{\Gamma}_Q\,d\Omega)\times\vec{K} \equiv \dfrac{(\vec{\Gamma}_Q\,d\Omega)\times\hat{r}_{QP}}{4\pi\,r_{QP}^2}$

which may be recognized (to within a multiplicative constant) as the familiar *Ampère's law of magnetic force* or the *Biot Savart law* for rotational fluid flows. Note that in the electrical case, the electric current is created by

the fundamental (electron) charge *traveling through* the conducting medium. The concept of a *traveling inducer* has thus long been a familiar one in that particular branch of physics. This concept can now be extended to *any* other physical situation involving a vector field.

It was indicated earlier that the vector field $\vec{F}(x,y,z)$ may be a *steady-state* field, invariant w.r.t. time, or it may be an *instantaneous snapshot* at a time t of a generalized *unsteady* field, varying in arbitrary fashion w.r.t. time *t*, as viewed from a stationary or *arbitrarily accelerating* frame of reference. In this generalized unsteady case, the *overall* influence induced at the point P at time t is the instantaneous sum (i.e. integral) of the elemental influences *induced* at point P by individual *inducer* elements contained in elemental volumes $d\Omega$ (or elemental surface areas dS) each located at some "running point" Q in Ω (or on its boundary S). The *value* of the intensity of each individual inducer is evaluated at that *same time* t. It is important to understand that for these *individual elemental influences*, there is no need to take account of any *time delay* associated with the finite propagation speed of "signals" originating from an individual elemental inducer at a "sending point" Q and arriving at a "receiving point" P. However, for any change occurring in the conditions at some region of the *boundary* S, which will then result in a progressive change in the values of the field elements, there will in general be a time delay associated with the resulting changes induced in the *overall* field induced at the point P.

This apparent anomaly can be explained by considering the *field* inducers, each contained at time t in an elemental volume $d\Omega$ somewhere in Ω, to represent the instantaneous local value of a time-varying distribution. For a generalized unsteady field, the author has introduced elsewhere the concept of *traveling inducers*. These are analogous in the general case to the *traveling electrons* indicated above for the specific case of the effect "induced" by an electric current.

For various physical fields of practical interest, three distinct cases can be identified:

(a) The field inducers are uniformly equal to zero throughout Ω, i.e. $\nabla_Q \cdot \vec{F}_Q = 0$ and $\nabla_Q \times \vec{F}_Q = 0$.

(b) *One* of the field inducers is uniformly equal to zero throughout Ω, i.e. $\nabla_Q \cdot \vec{F}_Q = 0$ or $\nabla_Q \times \vec{F}_Q = 0$.

(c) Both types of field inducers exist somewhere or everywhere in Ω, i.e. $\nabla_Q \cdot \vec{F}_Q \neq 0$ and $\nabla_Q \times \vec{F}_Q \neq 0$.

In *all three cases* we see from (25) that the field induced at P by the *field* inducers [which is *zero* in case (1) above] can be replaced *exactly* by the field induced at that same point by a composite surface distribution of inducers, *augmented*, when the point P lies in the *interior* of Ω, by the local value at P of the field $\vec{F}$:

(26) $$\iiint_\Omega [(\nabla_Q \cdot \vec{F}_Q)\vec{K} + (\nabla_Q \times \vec{F}_Q) \times \vec{K}]d\Omega \;\equiv\; T_P \vec{F}_P \;+\; \iint_S [(-\hat{n}_Q \cdot \vec{F}_Q)\vec{K} + (-\hat{n}_Q \times \vec{F}_Q) \times \vec{K}]dS \,.$$

We introduce the symbols σ and $\vec{\omega}$ defined at any point Q on the boundary S by: $\sigma_Q = -\hat{n}_Q \cdot \vec{F}_Q$ and $\vec{\omega}_Q = -\hat{n}_Q \times \vec{F}_Q$. By analogy with the *field* intensities Σ and $\vec{\Gamma}$ we can recognize σ as a *surface source* distribution and $\vec{\omega}$ as a surface distribution of "rotationality". In the case of a fluid flow, $\vec{\omega}$ is termed "surface vorticity". We can thus write (26) as:

(27) $$\iiint_\Omega [\Sigma_Q \vec{K} + \vec{\Gamma}_Q \times \vec{K}]d\Omega \equiv T_P \vec{F}_P + \iint_S [\sigma_Q \vec{K} + \vec{\omega}_Q \times \vec{K}]dS \,.$$

5. EVALUATION OF THE VECTOR FIELD INDUCED BY FIELD DISTRIBUTIONS

5.1 The Fundamental GENESIS Process

In the evaluation of *linear* fields by means of an integral method (i.e. problems in which the divergence and curl of the vector field external or internal to some boundary S are both uniformly equal to zero), we simply evaluate, point by point on the boundary, the value of distributions of source σ and/or rotationality $\vec{\omega}$ on that boundary S, which, in conjunction with some prescribed incident (unperturbed) field, satisfy the prescribed boundary conditions of the problem. Numerous so-called "panel methods" have been developed to solve such problems. In contrast, in the evaluation of *nonlinear* fields by means of an integral method (i.e. problems in which the divergence and/or curl of the vector field external or internal to some boundary are *nonzero*), we are also interested in the extra contribution induced by these *field* distributions. We have seen that the GENESIS Identity allows this extra contribution to the overall field to be reduced to the evaluation of the field induced by "equivalent" *surface* distributions of source and/or rotationality.

Suppose that the overall vector field in the *physical* problem is denoted by $\vec{V}(x,y,z)$. In a fluid-flow problem, this could represent the physical velocity vector; it could equally well represent a gravitational, electric, or magnetic field, or any other type of physical field. We can decompose this physical field into the sum of three contributions: $\vec{V} = \vec{V}_i + \vec{V}_\Omega + \vec{V}_S$, where:

- $\vec{V}_i$ is the *unperturbed* incident field (i.e. with no body present to perturb that field). In an infinite-domain problem, this can be considered as the contribution from a "boundary at infinity";
- $\vec{V}_\Omega$ is the perturbation field induced by the physically-defined, nonlinear *field* distributions of source $\Sigma = \nabla \cdot \vec{V}$ and/or rotationality $\vec{\Gamma} = \nabla \times \vec{V}$ in the domain Ω (the field $\vec{V}_\Omega$ induced by these distributions is defined by the l.h.s. of (27): $\vec{V}_\Omega = \iiint_\Omega [\Sigma\vec{K} + \Gamma \times \vec{K}]d\Omega$). In the purely *linear* case, of course, Σ and $\vec{\Gamma}$ and the perturbation field $\vec{V}_\Omega$ are all equal to zero. For a *nonlinear* problem the *values* of the distributions Σ and $\vec{\Gamma}$ will be defined in terms of the physical field $\vec{V}$ itself, as discussed earlier, and will be updated iteratively as the computation proceeds. The GENESIS Identity allows *this field contribution* $\vec{V}_\Omega$ to be reduced to the field induced by equivalent *surface* distributions of source σ_F and/or rotationality $\vec{\omega}_F$, augmented by a local vector $\vec{F}(x,y,z)$ for points which lie in the domain Ω.
- $\vec{V}_S \equiv \vec{V}_\sigma + \vec{V}_\omega$ is the field induced by boundary distributions of source σ and rotationality $\vec{\omega}$ located on the boundary *S*, with $\vec{V}_\sigma = \iint_S \sigma \vec{K} \ dS$ and $\vec{V}_\omega = \iint_S \vec{\omega} \times \vec{K}\, dS$. These distributions are also present in the *linear* problem, but will take a different value in the nonlinear problem because of the extra contribution to the overall field induced by the field distribution(s) Σ and/or $\vec{\Gamma}$ discussed above.

The general procedure involves numerically calculating the values of the surface distributions σ and $\vec{\omega}$ such that, in the presence of the incident field *and the field distributions*, the prescribed conditions at the boundary *S* are satisfied by the overall physical vector $\vec{V}$.

In the GENESIS computational algorithm developed, applied and described elsewhere in detail by the writer, the steps in the (iterative) nonlinear computation (considering here only the case in which no *discontinuities* are present in the field) may be summarized as follows:

(i) Compute the *values* of Σ and $\vec{\Gamma}$ at a number of representative points throughout Ω (more specifically, in those parts of the field in which these values are non-negligible by virtue of the presence of nonlinear effects). These *values* are obtained on the basis of the *physics* governing the physical field in question.

(ii) *Construct* a piecewise-continuous (fictitious) vector field $\vec{F}(x, y, z)$ whose divergence and curl are respectively equal to these computed values Σ and $\vec{\Gamma}$ at these same points.

(iii) Construct the equivalent *surface* functions $\sigma_Q = -\hat{n}_Q \cdot \vec{F}_Q$ and $\vec{\omega}_Q = -\hat{n}_Q \times \vec{F}_Q$ on the boundary *S*.

(iv) Interpret these equivalent surface functions as modifications to the *boundary conditions* of an *equivalent linear problem*, and solve this linear problem using, for example, a linear "panel method" or other suitable method.

Two very simple examples will be considered in Section 6, to illustrate the key concepts of the first three steps of this algorithm.

5.2 Evaluation of the Surface Integral Involving Rotationality

In certain cases, the "physics" of the physical situation will be such that the "primary" surface rotationality vector $\vec{\omega}$ in a linear or nonlinear problem, or correspondingly the "equivalent" surface rotationality $\vec{\omega}_F$ in a nonlinear problem, is *unidirectional*. That is, we can write $\vec{\omega} = \omega(x, y, z)\,\hat{\omega}$, where the unit vector $\hat{\omega}$ defining the direction of the axis of rotationality in the surface S is *constant*. This is always the case for two-dimensional problems in which the axis of rotationality points into the paper: $\hat{\omega} = \hat{j}$. The property of *local* unidirectionality (i.e. for the surface rotationality on each "panel" on surface *S*) also applies in many three-dimensional problems simply by virtue of the approximations that are made to facilitate the numerical solution. If we consider the vector field contribution $\delta\vec{V}_\omega$ induced by surface rotationality on some surface element *s*, we have: $\delta\vec{V}_\omega = \iint_s \vec{\omega} \times \vec{K}\, ds$. In the case where the *direction* $\hat{\omega}$ of the axis of rotationality is constant, this reduces to: $\delta\vec{V}_\omega = \hat{\omega} \times \iint_s \omega\, \vec{K}\, ds = \hat{\omega} \times \delta\vec{V}_\sigma$ where $\delta\vec{V}_\sigma$ is simply the vector field contribution induced by a surface *source* distribution $\sigma(x,y,z) = \omega(x,y,z)$ on the same surface *s*. We thus simply need to evaluate the field induced by this *source* distribution in order to immediately obtain the required field induced by the surface *rotationality*.

This concept can be readily extended to the more general case in which the axis of rotationality $\hat{\omega}$ is not constant over the area *s*. For example, if the axis of rotationality varies over the surface of the element *s*, we can represent this varying unit vector in terms of its direction cosines relative to each of the cartesian directions $\hat{i}, \hat{j}, \hat{k}$. We can write: $\hat{\omega} = \omega_x\, \hat{i} + \omega_y\, \hat{j} + \omega_z\, \hat{k}$. Inserting this in the above expression for the vector field contribution from element *s*, we have:

$$\delta\vec{V}_\omega = \hat{i} \times \iint_s (\omega\omega_x)\vec{K}\, ds + \hat{j} \times \iint_s (\omega\omega_y)\vec{K}\, ds + \hat{k} \times \iint_s (\omega\omega_z)\vec{K}\, ds = \hat{i} \times \delta\vec{V}_{x\sigma} + \hat{j} \times \delta\vec{V}_{y\sigma} + \hat{k} \times \delta\vec{V}_{z\sigma}$$

where, for example, $\delta\vec{V}_{x\sigma}$ is simply the vector field contribution induced by a surface *source* distribution $\sigma_x(x, y, z) = \omega(x, y, z)\ \omega_x(x, y, z)$ on the same surface element *s*.

The fact that the vector field induced by only surface *source* elements is required in a numerical formulation provides a significant simplification in the construction of a computer code, as well as a reduction in its execution time relative to formulations which do not take advantage of this property. The computation of the vector field induced by a single *source* "panel," at points located either at the midpoint of that panel ("self-influence") or at any other point, is a standard aspect of "panel codes" and will not be addressed here.

6. SOME SIMPLE EXAMPLES

6.1 Unit Cube with Uniform Field Source Distribution

Suppose the domain Ω is the interior of a unit cube with $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$. The horizontal unit vector $\hat{i}$ is oriented from left to right, $\hat{j}$ is oriented into the paper, and $\hat{k}$ is oriented vertically upwards.
We designate the leftmost vertical face of the cube (which is the plane $x = 0$) as the surface L; its unit normal, pointing into Ω, is thus $\hat{n}_L = \hat{i}$. We designate the rightmost vertical face of the cube (which is the plane $x = 1$) as the surface R, with unit normal $\hat{n}_R = -\hat{i}$, again pointing into Ω.
We designate the closest vertical face of the cube (which is the plane $y = 0$) as the surface C; its unit normal, pointing into Ω, is thus $\hat{n}_C = \hat{j}$. We designate the more distant vertical face of the cube (which is the plane $y = 1$) as the surface D, with unit normal $\hat{n}_D = -\hat{j}$, again pointing into Ω.
We designate the bottom horizontal face of the cube (which is the plane $z = 0$) as the surface B; its unit normal, pointing into Ω, is thus $\hat{n}_B = \hat{k}$. We designate the top horizontal face of the cube (which is the plane $z = 1$) as the surface T, with unit normal $\hat{n}_T = -\hat{k}$, again pointing into Ω.
The boundary S of the domain Ω is clearly the sum of the above six planar surfaces. (Note that in this example the boundary S contains angular discontinuities; however, the GENESIS Identity remains valid.)
The cube contains a uniform *source* density defined by $\Sigma = 1$, and *zero rotationality*. This fits into the general framework of category (b) above, and fulfills step (i) for this simple example.
To complete step (ii) above, we note that there exists an infinite number of distinct vector fields $\vec{F}$ which satisfy the conditions required here, namely $\nabla_Q \cdot \vec{F}_Q = 1$ and $\nabla_Q \times \vec{F}_Q = 0$. This will always be true for any situation encountered – *any* field $\vec{F}$ (from an infinite number of possible candidates) can be constructed whose divergence and curl match the desired functions. One field suitable for the present case is the horizontal linear field $\vec{F} = x\,\hat{i}$ (other candidates include the linear fields $\vec{F} = y\,\hat{j}$ and $\vec{F} = z\,\hat{k}$, as well as an infinite number of fields whose components are nonlinear functions of the coordinates x, y, and z).
To construct the equivalent surface functions in step (iii) above, using the field $\vec{F} = x\,\hat{i}$, we have on the left face L the surface source $\sigma_L = -\hat{n}_L \cdot \vec{F}_Q = -\hat{i} \cdot (x_L\,\hat{i}) = -x_L = 0$ since this surface is defined by $x = 0$.
Similarly on the left face L we have the surface rotationality $\vec{\omega}_L = -\hat{n}_L \times \vec{F}_Q = -\hat{i} \times (x_L\,\hat{i}) = 0$.
On the right face R the surface source $\sigma_R = -\hat{n}_R \cdot \vec{F}_Q = -(-\hat{i}) \cdot (x_R\,\hat{i}) = +x_R = 1$ since this surface is defined by $x = 1$. Similarly on face R we have surface rotationality $\vec{\omega}_R = -\hat{n}_R \times \vec{F}_Q = -(-\hat{i}) \times (x_R\,\hat{i}) = 0$, since $\hat{i} \times \hat{i} = 0$.
On the closest face C the surface source $\sigma_C = -\hat{n}_C \cdot \vec{F}_Q = -(\hat{j}) \cdot (x\hat{i}) = 0$ since $\hat{j} \cdot \hat{i} = 0$.
Similarly on face C we have the surface rotationality $\vec{\omega}_C = -\hat{n}_C \times \vec{F}_Q = -(\hat{j}) \times (x\hat{i}) = x\,\hat{k}$, since $-\hat{j} \times \hat{i} = \hat{k}$.
On the distant face D the surface source $\sigma_D = -\hat{n}_D \cdot \vec{F}_Q = -(-\hat{j}) \cdot (x\hat{i}) = 0$ since $\hat{j} \cdot \hat{i} = 0$.
Similarly on face D we have the surface rotationality $\vec{\omega}_D = -\hat{n}_D \times \vec{F}_Q = -(-\hat{j}) \times (x\,\hat{i}) = -x\,\hat{k}$.
On the top face T the surface source $\sigma_T = -\hat{n}_T \cdot \vec{F}_Q = -(-\hat{k}) \cdot (x\,\hat{i}) = 0$ since $\hat{k} \cdot \hat{i} = 0$.
Similarly on face T we have the surface rotationality $\vec{\omega}_T = -\hat{n}_T \times \vec{F}_Q = -(-\hat{k}) \times (x\,\hat{i}) = x\,\hat{j}$, since $\hat{k} \times \hat{i} = \hat{j}$.
On the bottom face B the surface source $\sigma_B = -\hat{n}_B \cdot \vec{F}_Q = -(\hat{k}) \cdot (x\hat{i}) = 0$ since $\hat{k} \cdot \hat{i} = 0$.
Similarly on face B we have the surface rotationality $\vec{\omega}_B = -\hat{n}_B \times \vec{F}_Q = -(\hat{k}) \times (x\,\hat{i}) = -x\,\hat{j}$, since $\hat{k} \times \hat{i} = \hat{j}$.

To summarize: The surface distributions equivalent to the field source $\Sigma = 1$ in the unit cube, for the choice $\vec{F} = x\,\hat{i}$ adopted here, are as follows:
The left face L carries no equivalent surface distributions: $\sigma_L = 0$ and $\vec{\omega}_L = 0$.
The right face R carries only a surface source distribution: $\sigma_R = 1$ and $\vec{\omega}_R = 0$.
The closest face C carries only a (linearly varying) surface rotationality: $\sigma_C = 0$ and $\vec{\omega}_C = x\,\hat{k}$.
The distant face D carries only a (linearly varying) surface rotationality: $\sigma_D = 0$ and $\vec{\omega}_D = -x\,\hat{k}$.
The top face T carries only a (linearly varying) surface rotationality: $\sigma_T = 0$ and $\vec{\omega}_T = x\,\hat{j}$.
The bottom face B carries only a (linearly varying) surface rotationality: $\sigma_B = 0$ and $\vec{\omega}_B = -x\,\hat{j}$.

By simple addition, we observe that the total amount of *surface source* is equal to *unity* (i.e. consisting solely of the unit surface source distribution $\sigma_R = 1$ on the unit area of the right face *R*.). This is equal to the total amount of source in the original *field* distribution (i.e. the unit field source distribution $\Sigma = 1$ in the unit cube). This (positive) source distribution on the right face induces a flux whose horizontal component is oriented from left to right at points external to the cube with $x > 1$; in contrast it induces a flux whose horizontal component is oriented from right to left at points in the interior of the cube, in fact at all points with $x < 1$.
Similarly, we observe that the total amount of *surface rotationality* on the surface of the cube is *zero* (i.e. the distributions on the top and bottom faces are equal but counter-rotating, and similarly for those on the closest and distant faces.) This is the same as in the original field rotationality in the cube (*zero*). Clearly, the *magnitude* of the equivalent surface rotationality at any point on surface *C*, *D*, *T* or *B* of the cube with coordinate x is equal to x. Each strip of infinitesimal width δx looks like a (square shaped) “ring vortex” or “loop electric current,” the plane of this “square ring” being located at x = constant. The surface density of this rotationality varies linearly between zero at the left end ($x = 0$) to unity at the right end ($x = 1$). Each of these “square rings” induces a flux whose horizontal component is oriented from right to left at any point in the interior of the cube – in fact for any point with $0 < y < 1$ and $0 < z < 1$. In contrast, for any point outside the extended cube (i.e. external to the region defined by $0 < y < 1$ and $0 < z < 1$) the horizontal component of the induced flux is oriented from left to right.
The jump in the induced field in proceeding across the unit surface source on the right face, from outside the cube ($x = 1^+$) to inside the cube ($x = 1^-$), is equal to a unit horizontal vector $-1\hat{i}$ oriented from right to left. Similarly the jump in the induced field in proceeding across the surface rotationality distribution on any of the faces *C*, *D*, *T* or *B* of the cube from outside to inside, at a point with coordinate x, is equal to a horizontal vector $-x\hat{i}$ oriented from right to left. This jump across any of the faces of the cube, due to the equivalent surface distributions, is clearly equal to the local value $-\vec{F}$ in the interior of the cube. This demonstrates that the field induced by these surface distributions (augmented by $+\vec{F}$ in the interior of the cube) is continuous across the surface of the cube. This corresponds to the field induced by the original *field* source distribution: this induced field is also continuous across the boundary of the cube.

6.2 Unit Cube with Zero Internal Field Distributions

We now consider the same unit cube as defined above, but this time the cube contains *zero* field source and *zero* field rotationality. The field induced by these “distributions” is obviously itself equal to zero. An infinite number of different field vectors $\vec{F}$ can be defined whose divergence and curl are both zero. One such vector is the horizontal unit vector $\vec{F} = 1\hat{i}$.

It is easy to construct the equivalent surface distributions for this case, using the same reasoning as that in Section 6.1 above:

The left face L carries only a surface *source* distribution: $\sigma_L = -1$ and $\vec{\omega}_L = 0$.
The right face R carries only a surface *source* distribution: $\sigma_R = 1$ and $\vec{\omega}_R = 0$.
The closest face C carries only a (uniform) surface *rotationality*: $\sigma_C = 0$ and $\vec{\omega}_C = 1\hat{k}$.
The distant face D carries only a (uniform) surface *rotationality*: $\sigma_D = 0$ and $\vec{\omega}_D = -1\hat{k}$.
The top face T carries only a (uniform) surface *rotationality*: $\sigma_T = 0$ and $\vec{\omega}_T = 1\hat{j}$.
The bottom face B carries only a (linearly varying) surface *rotationality*: $\sigma_B = 0$ and $\vec{\omega}_B = -1\hat{j}$.

The field induced by these surface distributions, *external* to the cube, is the same as that induced by the original field distributions, namely *zero*. The field induced by these surface distributions, in the *interior* of the cube, is in this case equal to the uniform field, $-\vec{F} = -1\hat{\imath}$. [In this particular case, the contribution to the vector field induced at the *center* of the cube by the distribution on *each* of the six faces of the cube is equal to $-\vec{F}/6 = -\frac{1}{6}\hat{\imath}$.] This demonstrates again that the field induced by these surface distributions (augmented by $+\vec{F} = +1\hat{\imath}$ in the interior of the cube) is continuous across the surface of the cube, and in this case equal to zero. This corresponds to the field induced by the original (*zero*) field distributions.
Note that although the field induced by any one of the surface distributions considered alone takes an *infinite* value at points in the immediate vicinity of the edges of the cube, these "infinities" cancel out when the overall surface distribution is considered. The overall field is *regular* everywhere inside and outside the cube, even in the vicinity of its edges and corners.
It is easy to see that there can be an infinite number of different distributions of source $\sigma_F = -\hat{n}\cdot\vec{F}$ and/or rotationality $\vec{\omega}_F = -\hat{n}\times\vec{F}$ on the surface of the cube — or indeed on the surface of *any* arbitrarily shaped closed surface *S* — corresponding to a different selection for the field $\vec{F}$ in the domain Ω (i.e. inside the region with boundary *S*), such that $\nabla\cdot\vec{F} = 0$ and $\nabla\times\vec{F} = 0$. All of these induce a field of *zero* outside the cube and a field equal to $-\vec{F}$ inside the body. Any such distribution located on the surface S will not perturb the external field (and thus will not violate the existing boundary conditions).
One interesting observation is worthy of note here: Suppose that there exists some (not necessarily uniform) incident field $\vec{F}_i$ defined throughout the entirety of space, but such that throughout the sub-region Ω bounded by the surface S we have $\nabla\cdot\vec{F}_i = 0$ and $\nabla\times\vec{F}_i = 0$. Then, locating a source distribution $\sigma = -\hat{n}\cdot\vec{F}_i$ and a rotationality distribution $\vec{\omega} = -\hat{n}\times\vec{F}_i$ on the surface S will not modify that field external to S (i.e. the *external* field will be totally unaffected and remain equal to the original incident field $\vec{F}_i$). However, the overall field in the *interior* of the sub-region Ω (i.e. the sum of the incident field and the interior field induced by these distributions on *S*) will now be precisely equal to zero: in effect, the distributions on S will totally *shield* the interior of Ω from the influence of the incident field. This phenomenon may be recognized as the "Faraday cage" effect familiar in the field of electromagnetism. This simple example indicates that in principle an analogous effect exists for *any* vector field, including gravity, fluid flows, etc.
There are in fact infinitely many different boundary distributions of source and rotationality which induce a *zero* field external to an arbitrary boundary S. This fact is of great value in the design of numerical methods for the solution of practical problems. It means that discretized distributions can be selected which also have optimal *numerical* properties. This topic is discussed at length in numerous publications in the field of boundary integral methods.